IEooc_Methods4_Exercise10: Matrix inversion methods in process-based LCA

Goal: Show that the two alternative matrix methods for constructing the supply chain of products: Leontief input-output and the matrix method by Heijungs and Suh (2002) are equivalent in the sense that the footprints calculated with those models are the same.

Problem setting

Environmental footprints of individual products cannot be measured directly and need to be constructed from process data. To make this construction reproducible and easy to carry out, a systematic approach is needed. LCA researchers realized that the linear process model commonly applied in the field allows the construction of the entire supply chain with a matrix inversion. Two approaches were developed: The first one is described in Heijungs and Suh (2002): "Computational Structure of Life Cycle Assessment" (Heijungs and Suh, 2002), and the second one is the Leontief input-output model, which was developed already in the first half of the 20th century but mostly applied to aggregated economic data before it found its way into LCA (Leontief, 1966). The Leontief approach uses the A-matrix of technical coefficients to link industry output x to inter-industry flows Z:

$$A = Z \cdot \hat{x}^{-1}, \qquad Z = A \cdot \hat{x} \qquad (1)$$

and the market balance to link industrial output x, inter-industry flows Z, and final demand y:

$$x = Z \cdot e + y \quad (2)$$

Here, ^ is the vector diagonalization (putting x on the diagonal of a else zero matrix), and e is a summation vector containing only 1s as elements. The emissions to the environment g are calculated with the per unit emissions/resource uptake *S*:

$$g = S \cdot x \quad (3)$$

Heijungs and Suh use a different approach: First, they define a "technology matrix" *F* of flows within the economy that contains an inventory of all recorded flows in the system. In their book, it is also called A, but the meaning is different from the Leontief A-matrix of technical coefficients. Leontief-A contains coefficients (each column is normalised to one unit of the corresponding row) and H&S-F (A) contains non-normalized flows. Inflows to processes get a negative sign and outflow a positive sign. Next to the technology matrix of inter-industry flows there is the so-called intervention matrix B, that records

1

resource uptake (input -> negative sign) and emissions (output -> positive sign). F (A) and B can be stacked to form the process description matrix P (Abb. 1).

The P matrix has some important features:

(1) Each row identifies one specific product (fuel, electricity, steel, … for F (A) and CO2, iron ore, … for B), and is recorded in the same unit across all columns. If a process does not have a certain input/output a 0 is entered.
(2) Each column represents a consistent/balanced inventory of a process, but the different columns can be inventoried independently of each other, at different scales.
(3) Like in Leontief IO, there is a strict 1:1 correspondence between rows (products) and columns (processes/industries).

$$\mathbf{P} = \left(\frac{\mathbf{A}}{\mathbf{B}} \right) = \begin{pmatrix} -2 & 100 \\ 10 & 0 \\ \hline 1 & 10 \\ 0.1 & 2 \\ 0 & -50 \end{pmatrix}$$

(4)

Figure 1: Process matrix P, consisting of inter-industry flows A (here: F) and emissions/resources B. Inflows to processes get a negative sign and outflow a positive sign. Resource uptake (input) has a negative sign and emissions (output) has a positive sign. Source: screenshot of equation (2.7) in Heijungs and Suh (2002). Rows: 2 products and 3 environmental extensions. Columns: 2 processes in 1:1 correspondence with 2 products.

The LCA is then formulated as the following problem: find a scaling vector s so that the s-weighted row sum of the products is equal to a given inhomogeneity y (f in the book), which is interpreted as final demand:

$$F \cdot s = y \quad \text{(5)}$$

That means, that each column $F_{\cdot j}$ is scaled by the corresponding element s_j and then summed up with the other scaled columns (scalar product), so that all inputs add up on the negative side and all outputs on the positive side. Their sum, the net output of the industrial system represented by F, must equal the exogenous demand y. If the database is completely populated, the scaling vector s can be determined via matrix inversion:

$$s = F^{-1} \cdot y \quad \text{(6)}$$

And the flows from/to the environment g via

$$g = B \cdot s \quad (7)$$

The final demand vector y is the same in both the Leontief and H&S cases. The matrices F and B (that together form P) are based on an independent inventory of the different industries, each at their own scale: e.g.: steel industry: all flows normalized per 1 ton of steel, electricity generation: total flows for 1 representative power station for one year, or car manufacturing: total flows for the entire sector for one year.

Task

Draw a system describing the situation, define the system variables, derive both supply chain models from the system definition, and show that both are equivalent in the sense that the footprints calculated with those models are the same! All considerations are to be done post-allocation, meaning that only processes that have exactly one output are to be considered, so that the technical coefficient matrix is square.

Sample solution

We take the system definition of the IO model found in **IEooc_Methods5_Exercise1:** Basics of Input-Output Modelling:

"Input-output analysis is based on a system description including both: transformation activities (industries, organisms) and distribution processes (markets). It contains four system variables: entry vector **v** ('value added'), total output vector **x**, final demand vector **y**, and matrix of intermediate **Z** (or A·X^)" (Fig. 2).
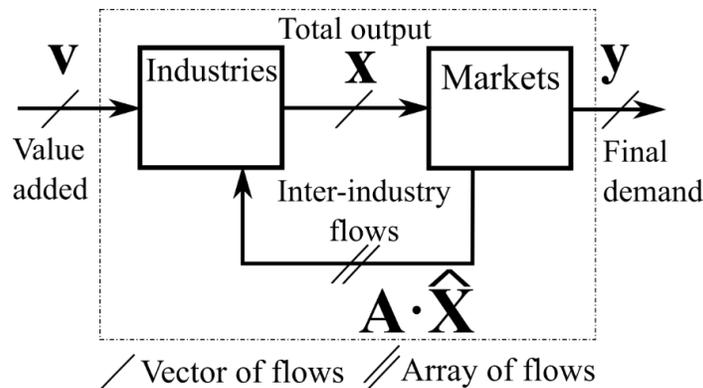

Figure 2: System structure of an IO model.

In Fig. 2 the emissions B are not shown. They are represented by a flow from/to the industries to/from the environment that does not enter the monetary balancing equations. The stressor matrix S is defined as the normalized emissions per unit of output x:

$$S = B \cdot \hat{x}^{-1} \quad (8)$$

The H&S process inventory can be written as

$$P = \begin{pmatrix} F \\ B \end{pmatrix} = \begin{pmatrix} \hat{x} - Z \\ B \end{pmatrix} \quad (9)$$

Which follows from H&S's convention that all inflows are negative and all outflows are positive. The value added v does not enter here as only industrial output is part of F. The H&S scaling equation can be written down in terms of the system variables in Fig. 2 and solved for s:

$$F \cdot s = y \qquad (\hat{x} - Z) \cdot s = y$$
$$(\hat{x} - Z \cdot \hat{x}^{-1} \cdot \hat{x}) \cdot s = y$$
$$(I - Z \cdot \hat{x}^{-1}) \cdot \hat{x} \cdot s = y$$
$$(I - A) \cdot \hat{x} \cdot s = y \qquad (10)$$
$$s = \hat{x}^{-1} \cdot (I - A)^{-1} \cdot y$$

And finally, the footprints:

$$g = B \cdot s$$
$$g = B \cdot \hat{x}^{-1} \cdot (I - A)^{-1} \cdot y \qquad (11)$$
$$g = S \cdot (I - A)^{-1} \cdot y$$

The Leontief case starts with the market balance, which is solved for x to obtain the Leontief IO model:

$$x = Z \cdot e + y$$
$$x = A \cdot \hat{x} \cdot e + y$$
$$x = A \cdot x + y$$
$$(I - A) \cdot x = y \qquad (12)$$
$$x = (I - A)^{-1} \cdot y$$

Footprints are determined by multiplying the stressor matrix S, which is defined as emissions per unit of x:

$$g = S \cdot x$$
$$g = S \cdot (I - A)^{-1} \cdot y \qquad (13)$$

Both H&S and Leontief supply chain models lead to the same footprints (equations 11 and 13). The main difference between the models is their construction: H&S uses internally consistent but disconnected process descriptions, whereas Leontief IO starts with a fully balanced description of the industrial system at scale. Because linear production functions are applied (definition of A!) to all industries/processes, both approaches lead to the same output x and emissions S·x for a given final demand y.

The calculations above are not a proof in the strict sense. They show how an underlying system definition (system variables as flows linked to processes) can help to understand how different footprint calculation methods are related to each other. Here we only considered the case where the H&S F matrix was determined from a balanced description of the industrial system. But it can also be quantified from internally consistent but disconnected process descriptions, which would then have to be reconciled (scaled) for form a balanced system description as shown in Fig. 2 to build an IO model.

References

Heijungs, R., Suh, S., 2002. Computational Structure of Life Cycle Assessment. Kluwer Academic Publications., Dordrecht, The Netherlands.

Leontief, W.W., 1966. Input-output economics. Oxford University Press, New York,.