Industrial Ecology Open Online Course (IEooc)

Part II Methods

Methods Part 5 (Input-output analysis)

http://www.teaching.industrialecology.uni-freiburg.de/

IEooc_Methods5_Exercise1: Basics of Input-Output Modelling

Sample solution

Goal: *Understand the mathematical basics of IO modelling, apply system structure of IO*

Input-output analysis is based on a system description including both: transformation activities (industries, organisms) and distribution processes (markets). It contains four system variables: entry vector **v** ('value added'), total output vector **x**, final demand vector **y**, and matrix of intermediate **Z** (or A·X^) (Fig. 1).

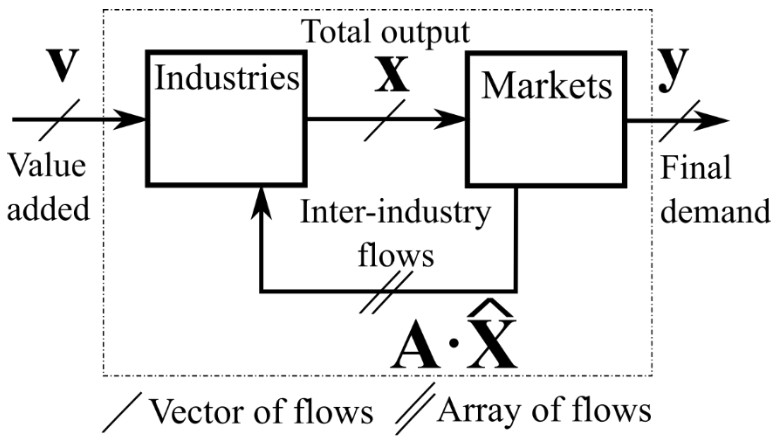

Figure 1: System structure of an IO model.

Exercises for model understanding:

1. **Make sure that the four system descriptions of the IO-model as shown in figure 1 are equivalent i.e. every description can be transformed into another without loss of information. How?**

➔ **a → b)** The data in the table of figure 1a account for both columns and rows. Consequently, you can extract the balance for rows and columns from this table:

$$x_i = \sum_j Z_{i\,j} + y_i$$

$$x_j = \sum_i Z_{i\,j} + v_j$$

They can then be written as matrix equation using ***e*** as a summation vector consisting of nothing but 1s.

$$x = Z \cdot e + y$$

$$x = Z^T \cdot e + v$$

b → a) By writing down the upper formula in tabular form you obtain the exact same structure as shown in a)

a → c) Every entry of Z, v, y and x is a flow between two nodes. Visualizing the non-explicit nodes in Fig. 1a you receive following image: Every row in table a) is the output of one node, a distribution node for product i to be exact. Every column of table a) is the input of one node namely a transformation node for making product j. All flows v_i originate from a node outside the system. All flows y_i end in a node outside the system. Within the system you can find node groups: j for transformation/production and i for goods distribution. A flows Z_{ij} go from one distribution node i to a production node j.

c and d) are equal since d) simply stacks the process and distribution nodes and then replicates them as vectors of nodes.

2. What do the upper and lower formulas of fig. 1b mean? Describe their meaning using your own words!

→ The upper formula is the market balance for the different goods produced. It implies that total production x either leaves the system to supply final consumers (y) or is used up by other industries (Z·e).
The lower formula is the balance of the industries that produce the different goods. Every single good has exactly one producer. The formula implies that the quantity of goods (x) produced is the sum of the consumed precursor products ($Z^T \cdot e$) and added value v.

3. How many processes are there in an IO-system with 18 industries and products?

→ 36. 18 industries and 18 markets.

4. How do you identify the matrix for production functions A using the given system variables? What exactly is the meaning of a matrix entry for A? What does an A column, A row stand for?

→short answer: A = Z x_hat^-1. Long answer: First you create a diagonal matrix from the outputs' vectors, **x**.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \rightarrow \hat{x} = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & x_n \end{pmatrix}$$

Then you take the inverse of x_hat:
Then you determine A by dividing the inter-industrial-flows with the associated output flows column by column:

$$\hat{x}^{-1} = \begin{pmatrix} 1/x_1 & 0 & \dots & 0 \\ 0 & 1/x_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1/x_n \end{pmatrix}$$

$$A = Z \cdot \hat{x}^{-1} = \begin{pmatrix} Z_{11}/x_1 & Z_{12}/x_2 & \dots & Z_{1n}/x_n \\ Z_{21}/x_1 & Z_{22}/x_2 & \dots & Z_{2n}/x_n \\ \dots & \dots & \dots & \dots \\ Z_{n1}/x_1 & Z_{n2}/x_2 & \dots & Z_{nn}/x_n \end{pmatrix}$$

A_ij shows the quantity of good i needed to produce one unit of good j, e.g., 12 kWh of electricity per kg aluminum. Column A_j describes the 'recipe for production' for good j, hence the quantity of all industrial goods required to produce one unit of j, e.g., electricity per ton of Al, aluminum oxide per ton of Al, fluorspar per ton of Al, etc.
Row A_i describes the relative quantities of good i required for the production of the different industrial goods produced, e.g., electricity per ton of Al, electricity per ton of paint, electricity per service unit, electricity per produced motor vehicle, etc.

5. **How do you obtain the matrix of market shares B using the given system variables? What exactly is the meaning of a matrix entry of B? What does a B column, B row stand for?**

Other than for A, for B the rows and not the columns are divided by the output vector ***x***. Analogous to A you can then find

$$B = \hat{x}^{-1} \cdot Z = \begin{pmatrix} Z_{11}/x_1 & Z_{12}/x_1 & \dots & Z_{1n}/x_1 \\ Z_{21}/x_2 & Z_{22}/x_2 & \dots & Z_{2n}/x_2 \\ \dots & \dots & \dots & \dots \\ Z_{n1}/x_n & Z_{n2}/x_n & \dots & Z_{nn}/x_n \end{pmatrix}$$

B_ij denotes the inter-industrial flow of good i to industry j per output xi. B_ij hence describes the share of consumption of industry j in the total production of good i.

Column B_j describes the consumption share of industry j in relation to total production of all goods. For example, Al industry consumes 3 % of total electricity, 98 % of total aluminum oxide, 65 % of total fluorspar, etc.

Row B_i defines the share of all industries in the consumption of good i, for example 3 % of electricity goes towards Al industry, 2 % towards steel industry, 5 % to agriculture, etc.

6. **Prove that $A = \hat{X}B\hat{X}^{-1}$ and $B = \hat{X}^{-1}A\hat{X}$! What do these formulas stand for?**

$$\hat{x} \cdot \hat{x}^{-1} = I$$

$$A = Z \cdot \hat{x}^{-1} = I \cdot Z \cdot \hat{x}^{-1} = \left(\hat{x} \cdot \hat{x}^{-1}\right) \cdot Z \cdot \hat{x}^{-1} = \hat{x} \cdot \left(\hat{x}^{-1} \cdot Z\right) \cdot \hat{x}^{-1} = \hat{x} \cdot B \cdot \hat{x}^{-1}$$

We use the equation that defines A and insert something called 'intelligent 1'.

The second relationship for B is a direct result of the first by multiplying x_hat^-1 from the left and x_hat from the right.

First, the formulas indicate that the information content is the same for A and B. Both matrices can be tranformed into one another using the scaling vector x.

Further you can discover based on the given formulas that A and B can be generated from one another through the application of the normalization matrix x_hat.

7. Determine the formula to calculate Leontief Inverse L using given system variables! What does a single entry for L stand for? What's the difference between A and L?

$$L = (I - A)^{-1} = \left(I - Z \cdot \hat{x}^{-1}\right)^{-1}$$

We want to find the relationship between L and the system variables x, y, v, and Z. This we can do by replacing A with the definition of A.

L is the matrix for the demand-driven IO model. The element L_ij determines how much of good i needs to be produced in the upstream chain to deliver one unit of good j to the final consumer. A contains data describing individual industrial processes; it also defines exactly one step of the supply chain of all products. L on the other hand includes all (infinite number of) steps of the entire supply chain of all products. It does not describe individual industrial processes only but a complex industrial network.

8. Determine the formula to calculate Gosh inverse G using given system variables! What does a single entry for G stand for? What's the difference between B and G?

$$G = \left(I - B^T\right)^{-1} = \left(I - Z^T \cdot \hat{x}^{-1}\right)^{-1}$$

We want to find the relationship between G and the system variable x, y, v, and Z. This we can do by replacing B with the definition of B.
G is the matrix for the supply-driven IO model. An element G_ij defines how many of good i can be produced in the whole economy given one unit of added value in industrial sector j. B defines data for individual, simultaneously existing markets and also defines exactly one step of the distribution of goods. G on the other hand includes all (infinite number of) steps of the distribution of goods and it does not describe individual markets only but a complex industrial network.

9. **Describe two specific examples for when to use Leontief and Gosh matrices. During the cold war era the hypothesis was that the Leontief model represents the capitalistic economic system while the Gosh model represents the socialistic planned economic system. How did people come up with this idea? Do you agree/disagree?**

Example Leontief: I buy a bottle of apple juice and calculate the output of the different industrial sectors (electricity supply, transport, refinery, …) of the bottle's upstream production chain using L.

Or calculating the total global economy output which was produced directly and indirectly for the German final demand of goods using the total demand of Germany, 2015, separated into different goods.

Example Gosh: I assume certain added value in a specific industrial sector and then calculate the distribution of added value among different goods in the economy using the Gosh model.

Or: There are several growth-limiting nutrients in an eco-system, e.g., nitrate, and I determine their impact on different species and subsequent changes in the whole food chain using the Gosh model. The different species in the eco-system represent the processes and the number of organisms is the output of the processes, which are either consumed by other organisms (Z) or increase the inventory of the own species (y).

People used to believe that the symmetry of the two models represent the at that time rival economic systems. The Leontief IO model is demand-driven and was applied in the USA during the war for example to improve planning of the industrial system dealing with the changes caused by the new war economy. On the other hand, planned economies stipulate the distribution of goods, or the consumption share, of every single industrial sector, hence people believed that the economic system of Gosh should be applied.

The system of stipulated market shares is unrealistic. Its application would mean that for each increase of production of x_i of 10 %, for example, all other sectors that obtain some fraction of x_i would receive 10 % more, no matter if they need it to produce output or not. That means the model assumes that with increasing tire production the ceteris paribus clause applies and hence automatically an increase of motor vehicle production can be observed. Consequently, based on the Gosh model's unrealistic production function assumptions it usually does not get applied to investigate changes in the system.

Actually, even countries that applied planned economy used only the Leontief model.

The Ghosh model is useful, however, to trace portions of value added through the economy. It can be applied as attributional model to trace the fate of capital services or subsidies, or of natural resources for a model in physical units.

Numerical exercise:

In the table 'Raw Data' of the work book 'IEooc_Methods5_Exercise1_IO_Basics.xlsx' provided you can find data for a simple IO system with four industries and markets. The cells used to determine each step of the solution are already marked.

1. Complete the data base (blue): Calculate the value added of the processes to obtain the four system variables to meet the requirements for the market and process balance sheet!
2. Development of the IO-model (green): Calculate the A and L matrices for the given data! (Excel function: MMULT for matrix multiplication, MTRANS to transpose the matrices, MINV for matrix inversion. Caution! The results must be calculated using only matrix formulas. https://support.office.com/en-us/article/Guidelines-and-examples-of-array-formulas-7d94a64e-3ff3-4686-9372-ecfd5caa57c7)
3. What's the output vector x for given final demand y according to Leontief model? How big is the overall input v for the Leontief model and why is the overall input exactly that big?

➔ **Solution and comments: see excel file IEooc_Methods5_Exercise1_IO_Basics_Solution.xlsx**